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A $(1+1)$-Dimensional Reduced Model of Mesons

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Abstract

We propose an extension of 't Hooft's large-N_c light-front QCD in two dimensions to include helicity and physical gluon degrees of freedom, modelled on a classical dimensional reduction of four dimensional QCD. A non-perturbative renormalisation of the infinite set of coupled integral equations describing boundstates is performed. These equations are then solved, both analytically in a phase space wavefunction approximation and numerically by discretising momenta, for (hybrid) meson masses and (polarized) parton structure functions.

1 Introduction

't Hooft suggested a two-dimensional model of mesons [1] which is a tractable non-abelian gauge theory sharing many qualitative properties with true QCD in four dimensions. The keys to its solution involved the use of the large N_c limit and light-front quantisation in light-front gauge. Since it is two dimensional, it contains no spin or physical gluons however; these are perhaps the least well understood structure of true hadrons. One may retain a remnant of such transverse features, whilst keeping the kinematics 1+1-dimensional, by picking two (arbitrary) space dimensions $x_\perp = \{x^1, x^2\}$ and considering only the zero modes

$$\partial_{x_\perp} A_\mu = \partial_{x_\perp} \Psi = 0 \tag{1}$$

of the gauge and quark fields. This may well be an appropriate approximation in some high energy scattering processes when transverse momenta are relatively damped. With a view to employing this fact, in this paper we construct the mesons appearing in a $1+1$-dimensional reduced model specified by imposing (1) at the classical level, which will appear as asymptotic or exchanged states (in the Regge sense) for scattering processes in this model.[1] These states need not be similar to the true hadrons involved in high energy scattering, since solving the boundstate problem for only co-linear ($k_\perp = 0$) quarks and gluons is not the same as solving the full boundstate problem and then going to a regime where $k_\perp$ is relatively small. However, the results presented here and in ref.[2] encourage us that the hadrons reduced in this way share many qualitative properties with true hadrons, even those Lorentz-invariant ones for which $k_\perp = 0$ cannot be kinematically justified, and therefore are of interest in their own right.

In the large-N_c light-front formalism the reduced hadrons satisfy infinite sets of coupled boundstate integral equations which, we argue, are rendered finite equation-by-equation by retaining parton self-energies. Solutions to the equations are obtained analytically in a phase space wavefunction approximation and numerically by truncating and discretising them. These solutions yield both meson and hybrid meson masses, as well as (polarized) quark and gluon structure functions.

2 Dimensional Reduction.

We start from $SU(N_c)$ gauge theory in 3+1-dimensions ($\mu \in \{0,1,2,3\}$) with one flavour of quarks

$$S = \int d^4x \left[-\frac{1}{4\tilde{g}^2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \mathrm{i}\bar{\Psi}\gamma^\mu_{(4)} D_\mu \Psi - m\bar{\Psi}\Psi \right] \tag{2}$$

[1] We have performed a detailed analysis of the corresponding pure glue states in a recent paper [2], building upon earlier work [3]. Similar work, with a truncated light-front Fock space, has recently appeared for $N_c = 3$ [4]

in the Weyl representation

$$\gamma^0_{(4)} = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \ , \ \gamma^i_{(4)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ . \tag{3}$$

Imposing (1), one finds an effectively two-dimensional gauge theory of adjoint scalars and fundamental Dirac spinors with action

$$\begin{aligned} S &= \int dx^0 dx^3 \ \Big\{ -\frac{1}{4g^2}\mathrm{Tr}[F_{ab}F^{ab}] + \frac{\mathrm{i}}{\sqrt{2}}(\bar{u}\gamma^a_{(2)}D_a u + \bar{v}\gamma^a_{(2)}D_a v) + \frac{m}{\sqrt{2}}(\bar{u}v + \bar{v}u) \\ &+\mathrm{Tr}\left[-\frac{1}{2}\bar{D}_a\phi_\rho\bar{D}^a\phi^\rho - \frac{tg^2}{4}[\phi_\rho,\phi_\sigma][\phi^\rho,\phi^\sigma] + \frac{1}{2}m_0^2\phi_\rho\phi^\rho \right] \\ &-\frac{sg}{\sqrt{2}}(\bar{u}(\phi_1 + \mathrm{i}\gamma^5_{(2)}\phi_2)u - \bar{v}(\phi_1 - \mathrm{i}\gamma^5_{(2)}\phi_2)v) \Big\} \ , \end{aligned} \tag{4}$$

where a and $b \in \{0,3\}$, $\rho \in \{1,2\}$, $\gamma^0_{(2)} = \sigma^1$, $\gamma^3_{(2)} = \mathrm{i}\sigma^2$, $\gamma^5_{(2)} = \mathrm{i}\sigma^1\sigma^2$, $\phi_\rho = A_\rho/g$, $g^2 = \tilde{g}^2/\int dx^1 dx^2$, $\bar{D}_a = \partial_a + \mathrm{i}[A_a, .]$, $D_a = \partial_a + \mathrm{i}A_a$. The two-component spinors u and v are related to Ψ by

$$2^{1/4}\Psi\sqrt{\int dx^1 dx^2} = \begin{pmatrix} u_{R+} \\ u_{L+} \\ u_{L-} \\ u_{R-} \end{pmatrix} \ , \ u = \begin{pmatrix} u_{L+} \\ u_{R+} \end{pmatrix} \ , \ v = \begin{pmatrix} u_{L-} \\ u_{R-} \end{pmatrix} \ . \tag{5}$$

The suffices L (R) and $+$ ($-$) in (5) represent Left (Right) movers and $+ve$ ($-ve$) helicity. Thus u, v, ϕ_1, ϕ_2 represent the transverse polarisations of the $3+1$ dimensional quarks and gluons. Since the dimensional reduction procedure treats space asymmetrically, we have allowed the gauge couplings in transverse directions to be different from that in the longitudinal direction in general, through dimensionless parameters s and t (canonically $s = -t = 1$), and have added a bare mass m_0 for the ϕ fields; both these modifications can occur due to loss of transverse local gauge transformations. The fact that the reduction to colinearity has been performed at the classical level, avoiding the singularities of the corresponding procedure performed after quantisation, emphasizes that this is a *model*. The couplings and masses are therefore left as adjustable parameters; for example, the s and t couplings will control the magnitude of helicity splittings in the spectrum.

The action (4) is a combination of gauged fundamental [1] and adjoint matter [3] representations with further Yukawa [5, 6] and matrix scalar [7] interactions, each of which have been individually studied by light-front quantisation before. We therefore only sketch the construction of boundstate equations. These equations will be valid for the modes of the theory at $N_c = \infty$ with non-zero light-front momenta $k^+ = (k^0 + k^3)/\sqrt{2} \neq 0$. Zero k^+ modes in reduced models have been discussed in refs.[4, 8] for example, and while there is no evidence that they affect the spectrum in the two-dimensional large-N_c theory, one should bear in mind their omission in this initial investigation.

In the naive light-front gauge $A_- = (A_0 - A_3)/\sqrt{2} = 0$, the fields A_+ and $u_{L\pm}$ are non-propagating in light-front time $x^+ = (x^0 + x^3)/\sqrt{2}$, satisfying constraint equations

$$
\begin{aligned}
0 &= \mathrm{i}\partial_- u_{L+} + \frac{m}{\sqrt{2}} u_{R-} - sgB_+ u_{R+} && (6)\\
0 &= \mathrm{i}\partial_- u_{L-} + \frac{m}{\sqrt{2}} u_{R+} + sgB_- u_{R-} && (7)\\
0 &= \partial_-^2 A_+ - g^2 J^+ && (8)
\end{aligned}
$$

where the helicity fields and longitudinal momentum current are

$$
\begin{aligned}
B_\pm &= (\phi_1 \pm \mathrm{i}\phi_2)/\sqrt{2} && (9)\\
J^+_{ij} &= \mathrm{i}[B_-, \partial_- B_+]_{ij} + \mathrm{i}[B_+, \partial_- B_-]_{ij} + u_{R+i}u^*_{R+j} + u_{R-i}u^*_{R-j}\ . && (10)
\end{aligned}
$$

In eq(10) we have explicitly shown the colour indices $i, j = 1, \ldots, N_c$. Eliminating the constrained fields in favour of non-local instantaneous interactions gives the light-front energy and momentum

$$
\begin{aligned}
P^- &= \int dx^- \left[\frac{tg^2}{2}(B_{+ij}B_{+jk}B_{-kl}B_{-li} - B_{+ij}B_{-jk}B_{+kl}B_{-li}) - \frac{g^2}{2} J^+_{ij}\frac{1}{\partial_-^2}J^+_{ji} + \frac{1}{2}m_0^2 B_{+ij}B_{-ji} \right.\\
&\quad - \frac{\mathrm{i}m^2}{2}\left(u^*_{-i}\frac{1}{\partial_-}u_{-i} + u^*_{+i}\frac{1}{\partial_-}u_{+i}\right) - \mathrm{i}s^2g^2\left(u^*_{+i}B_{-ij}\frac{1}{\partial_-}B_{+jk}u_{+k} + u^*_{-i}B_{+ij}\frac{1}{\partial_-}B_{-jk}u_{-k}\right)\\
&\quad \left. - \frac{\mathrm{i}sgm}{\sqrt{2}}(u^*_{+i}[\partial_-^{-1}, B_{-ij}]u_{-j} + u^*_{-i}[B_{+ij}, \partial_-^{-1}]u_{+j}) \right] && (11)\\
P^+ &= \int dx^- \left[2\partial_- B_{+ij}\partial_- B_{-ji} + \mathrm{i}(u^*_{+i}\partial_- u_{+i} + u^*_{-i}\partial_- u_{-i})\right] && (12)
\end{aligned}
$$

where the right-mover subscript R on u has been dropped. The fourier modes defined at $x^+ = 0$ by[2]

$$
\begin{aligned}
u_{\pm i} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty dk^+ \left(b_{\pm i}(k^+)\mathrm{e}^{-ik^+x^-} + d^\dagger_{\mp i}(k^+)\mathrm{e}^{ik^+x^-}\right) && (13)\\
B_{\pm ij} &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left(a_{\mp ij}(k^+)\mathrm{e}^{-ik^+x^-} + a^\dagger_{\pm ji}(k^+)\mathrm{e}^{ik^+x^-}\right) && (14)
\end{aligned}
$$

satisfy

$$
\{b_{\alpha i}(k^+), b^\dagger_{\beta j}(\tilde{k}^+)\} = \delta(k^+ - \tilde{k}^+)\delta_{ij}\delta_{\alpha\beta}\ ,\ [a_{\alpha ij}(k^+), a^\dagger_{\beta kl}(\tilde{k}^+)] = \delta(k^+ - \tilde{k}^+)\delta_{ik}\delta_{jl}\delta_{\alpha\beta} \qquad (15)
$$

when the canonical equal-x^+ relations hold

$$
\{u^*_{\alpha i}(x^-), u_{\beta j}(\tilde{x}^-)\} = \delta(x^- - \tilde{x}^-)\delta_{ij}\delta_{\alpha\beta}\ ,\ [\phi_{\alpha ij}(x^-), \partial_-\phi_{\beta kl}(\tilde{x}^-)] = \frac{\mathrm{i}}{2}\delta(x^- - \tilde{x}^-)\delta_{il}\delta_{jk}\delta_{\alpha\beta}\ , \qquad (16)
$$

[2] symbol † is the quantum version of complex conjugate $*$ and does not act on colour indices.

where α and $\beta \in \{+,-\}$ label helicity. The light-front Fock vacuum is well known to be trivial, satisfying $: P^+ : |0> =: P^- : |0> = 0$, so Fock states built with the oscillators (15) have a direct partonic interpretation. In the large-N_c limit with $g^2 N_c$ fixed, the finite solutions to the mass-shell condition $M^2\Psi = 2P^+P^-\Psi$ are formed from singlet linear combinations of the Fock states under residual x^--independent gauge transformations

$$
\begin{aligned}
\Psi \;=\; & \sum_{n=2}^{\infty} \int_0^{P^+} dk_1^+ dk_2^+ \ldots dk_n^+ \; \delta\left(k_1^+ + k_2^+ + \cdots + k_n^+ - P^+\right) \times \\
& \frac{f_{\alpha\beta\gamma\ldots\eta\delta}(k_1^+,\ldots,k_n^+)}{\sqrt{N_c^{n-1}}} d^\dagger_{\alpha i}(k_1^+) a^\dagger_{\beta ij}(k_2^+) a^\dagger_{\gamma jk}(k_3^+) \ldots a^\dagger_{\eta lp}(k_{n-1}^+) b^\dagger_{\delta p}(k_n^+)|0>, \qquad (17)
\end{aligned}
$$

Only these singlets satisfy the quantum version of the zero mode of eq(8), $\int dx^- : J^+ : \Psi = 0$. They are also eigenstates of the total helicity

$$
h = \int_0^{P^+} dk^+ \sum_\alpha \mathrm{sgn}(\alpha) \left[\frac{1}{2}(b^\dagger_{\alpha i} b_{\alpha i}(k^+) + d^\dagger_{\alpha i} d_{\alpha i}(k^+)) + a^\dagger_{\alpha ij} a_{\alpha ij}(k^+)\right] , \qquad (18)
$$

forming degenerate pairs $\pm h$ when $h \neq 0$, and total momentum P^+. It remains therefore to find the coefficients f which diagonalize P^-.

3 Boundstate Equations.

Substituting the mode expansions (13)(14) into $P^\pm$ (11)(12) and discarding infinite additive constants but *not* quadratic terms resulting from normal ordering one finds

$$
P^+ = \int_0^\infty dk\; k\left(b^\dagger_{\alpha i} b_{\alpha i}(k) + d^\dagger_{\alpha i} d_{\alpha i}(k) + a^\dagger_{\alpha ij} a_{\alpha ij}(k)\right) \qquad (19)
$$

and $P^- = P^-_{\text{trans.}} + P^-_{\text{long.}} + P^-_{\text{glue}}$, where

$$
\begin{aligned}
P^-_{\text{trans.}} \;=\; & \frac{s^2 g^2}{2\pi} \int_0^\infty dk_1 dk_2 dk_3 dk_4 \left\{ \frac{\delta(k_1+k_2-k_3-k_4)}{\sqrt{k_1 k_3}(k_1+k_2)} \times \right. \\
& \left(a^\dagger_{+ik}(k_3) b^\dagger_{-k}(k_4) a_{+il}(k_1) b_{-l}(k_2) + d^\dagger_{+k}(k_4) a^\dagger_{-kj}(k_3) d_{+l}(k_2) a_{-lj}(k_1)\right) \\
& + \frac{\delta(k_1+k_2+k_3-k_4)}{\sqrt{k_1 k_2}(k_2+k_3)} \times \\
& \left(b^\dagger_{+i}(k_4) a_{+ik}(k_1) a_{-kl}(k_2) b_{+l}(k_3) + d^\dagger_{+i}(k_4) d_{+l}(k_3) a_{-lk}(k_2) a_{+ki}(k_1)\right. \\
& \left. + a^\dagger_{+ik}(k_1) a^\dagger_{-kl}(k_2) b^\dagger_{+l}(k_3) b_{+l}(k_4) + d^\dagger_{+i}(k_3) a^\dagger_{-ik}(k_2) a^\dagger_{+kl}(k_1) d_{+l}(k_4)\right) \\
& \left. + \quad (\text{interchange } + \leftrightarrow -) \qquad \right\} \\
+ \; & \frac{msg}{2\sqrt{\pi}} \int_0^\infty dk_1 dk_2 dk_3 \left\{ \delta(k_1+k_2-k_3) \frac{1}{\sqrt{k_1}} \left(\frac{1}{k_2} - \frac{1}{k_3}\right) \times \right.
\end{aligned}
$$

$$
\begin{aligned}
& \Big(b^\dagger_{+i}(k_3) a_{+ij}(k_1) b_{-j}(k_2) - d^\dagger_{+i}(k_3) d_{-j}(k_2) a_{+ji}(k_1) \\
& + a^\dagger_{+ij}(k_1) b^\dagger_{-j}(k_2) b_{+i}(k_3) - d^\dagger_{-i}(k_2) a^\dagger_{+ij}(k_1) d_{+j}(k_3) \Big) \\
& - \quad (\text{interchange } + \leftrightarrow -) \qquad \Big\} \\
+ \quad & \frac{1}{2} m^2 \int_0^\infty \frac{dk}{k} \left(d^\dagger_{\alpha i}(k) d_{\alpha i}(k) + b^\dagger_{\alpha i}(k) b_{\alpha i}(k) \right) \\
+ \quad & \frac{s^2 g^2 N_c}{2\pi} \int_0^\infty \frac{dk_1 dk_2}{k_1(k_2 - k_1)} \left(b^\dagger_{\alpha i}(k_2) b_{\alpha i}(k_2) \right) \\
+ \quad & \frac{s^2 g^2 N_c}{2\pi} \int_0^\infty \frac{dk_1 dk_2}{k_1(k_2 + k_1)} \left(d^\dagger_{\alpha i}(k_2) d_{\alpha i}(k_2) \right), \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
P^-_{\text{long.}} \quad = \quad & \frac{g^2}{2\pi} \int_0^\infty dk_1 dk_2 dk_3 dk_4 \; \Big\{ \frac{\delta(k_1 + k_2 - k_3 - k_4)}{(k_2 - k_4)^2} \, d^\dagger_{\alpha i}(k_3) b^\dagger_{\beta j}(k_4) d_{\alpha i}(k_1) b_{\beta j}(k_2) \\
- \quad & \delta(k_1 + k_2 - k_3 - k_4) \, \frac{(k_1 + k_3)}{2\sqrt{k_1 k_3}(k_2 - k_4)^2} \times \\
& \Big(a^\dagger_{\alpha ij}(k_3) b^\dagger_{\beta j}(k_4) a_{\alpha ik}(k_1) b_{\beta k}(k_2) + d^\dagger_{\alpha j}(k_4) a^\dagger_{\beta ji}(k_3) d_{\alpha k}(k_2) a_{\beta ki}(k_1) \Big) \\
+ \quad & \delta(k_1 + k_2 + k_3 - k_4) \, \frac{k_1 - k_2}{2\sqrt{k_1 k_2}(k_1 + k_2)^2} (1 - \delta_{\alpha\gamma}) \times \\
& \Big(a^\dagger_{\alpha ij}(k_1) a^\dagger_{\gamma jk}(k_2) b^\dagger_{\beta k}(k_3) b_{\beta i}(k_4) + d^\dagger_{\beta k}(k_3) a^\dagger_{\alpha kj}(k_2) a^\dagger_{\gamma ji}(k_1) d_{\beta i}(k_4) \\
& + b^\dagger_{\beta i}(k_4) a_{\alpha ij}(k_1) a_{\gamma jk}(k_2) b_{\beta k}(k_3) + d^\dagger_{\beta i}(k_4) d_{\beta k}(k_3) a_{\alpha kj}(k_2) a_{\gamma ji}(k_1) \Big) \Big\} \\
+ \quad & \frac{g^2 N_c}{4\pi} \int_0^\infty dk_1 dk_2 \left(\frac{1}{(k_1 - k_2)^2} - \frac{1}{(k_1 + k_2)^2} \right) \times \\
& \qquad \Big(d^\dagger_{\beta i}(k_1) d_{\beta i}(k_2) + b^\dagger_{\beta i}(k_1) b_{\beta i}(k_2) \Big), \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
P^-_{\text{glue}} \quad = \quad & \frac{g^2}{2\pi} \int_0^\infty \frac{dk_1 dk_2 dk_3 dk_4}{\sqrt{k_1 k_2 k_3 k_4}} \; \Big\{ \delta(k_1 + k_2 - k_3 - k_4) \times \\
& \Big[(t - A_2) \, a^\dagger_{+ij}(k_3) a^\dagger_{+jk}(k_4) a_{+il}(k_1) a_{+lk}(k_2) \\
& + (t + A_1) \, a^\dagger_{+ij}(k_3) a^\dagger_{-jk}(k_4) a_{-il}(k_1) a_{+lk}(k_2) \\
& + (A_1 + A_2 - 2t) \, a^\dagger_{+ij}(k_3) a^\dagger_{-jk}(k_4) a_{+il}(k_1) a_{-lk}(k_2) \; \Big] \\
+ \quad & \delta(k_1 + k_2 + k_3 - k_4) \Big[\; (B_1 + B_2 - 2t) \times \\
& \Big(a^\dagger_{+ij}(k_1) a^\dagger_{-jk}(k_2) a^\dagger_{+kl}(k_3) a_{+il}(k_4) + a^\dagger_{+ij}(k_4) a_{+il}(k_1) a_{-lk}(k_2) a_{+kj}(k_3) \Big) \\
& + (t + B_1) \; \Big(a^\dagger_{+ij}(k_1) a^\dagger_{-jk}(k_2) a^\dagger_{-kl}(k_3) a_{-il}(k_4) + a^\dagger_{-ij}(k_4) a_{+il}(k_1) a_{-lk}(k_2) a_{-kj}(k_3) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+(t+B_2)\ \left(a^\dagger_{+ij}(k_1)a^\dagger_{+jk}(k_2)a^\dagger_{-kl}(k_3)a_{+il}(k_4)+a^\dagger_{+ij}(k_4)a_{+il}(k_1)a_{+lk}(k_2)a_{-kj}(k_3)\right)\ \Big] \\
&+\ [\quad (\text{interchange } + \leftrightarrow -) \qquad] \ \Big\}\ +\ \frac{g^2N_c}{\pi}\int_0^\infty \frac{dk_2}{k_2}\int_0^{k_2} dk_1 \frac{1}{(k_2-k_1)^2}a^\dagger_{\alpha ij}(k_2)a_{\alpha ij}(k_2) \\
&+\quad \frac{g^2N_c(1-t/2)}{4\pi}\int_0^\infty \frac{dk_1dk_2}{k_1k_2}a^\dagger_{\alpha ij}(k_2)a_{\alpha ij}(k_2), \qquad (22)
\end{aligned}
$$

where the coefficients A_1, A_2, B_1, B_2 are defined by

$$
A_1 = \frac{(k_2-k_1)(k_4-k_3)}{4(k_1+k_2)^2}, \qquad A_2 = \frac{(k_1+k_3)(k_2+k_4)}{4(k_4-k_2)^2}, \tag{23}
$$

$$
B_1 = \frac{(k_1-k_2)(k_3+k_4)}{4(k_1+k_2)^2}, \qquad B_2 = \frac{(k_1+k_4)(k_3-k_2)}{4(k_2+k_3)^2}. \tag{24}
$$

A simplification of large N_c is that $2 \leftrightarrow 2$ or $1 \leftrightarrow 3$ interactions occur only between *neighboring* partons in the colour contraction (17). Pair production of quarks, but not gluons, is absent. The last integral in (21), last two integrals in (20), and last two integrals in (22) are x^+-instantaneous self-energy terms (refered to as s,t, and g self-energy hereafter) resulting from normal ordering, which lead to a finite spectrum if retained. This is most easily demonstrated by projecting $M^2\Psi$ onto individual Fock states to derive integral equations for the coefficients f. One obtains infinite towers of successively coupled equations that are too long to write in full detail here, so we exhibit salient properties only.

3.1 Longitudinal.

In this section we turn off the transverse couplings $s = t = 0$, so that only the number n of partons is significant rather than their individual helicities. If we also neglect the longitudinal processes which change the number of partons, which is in fact an excellant approximation for low eigenvalues M [3], then for $f_n(x_1,\ldots,x_n)$, where $x_i = k_i^+/P^+$ labels momentum fraction, one finds

$$
\begin{aligned}
\frac{M^2\pi}{g^2N_c}f_n \ &= \ \frac{m_B^2\pi}{g^2N}[\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}]f_n + \frac{m^2\pi}{g^2N}[\frac{1}{x_1}+\frac{1}{x_n}]f_n + \delta_{n2}\int_0^1 dy\left\{\frac{f_2(x_1,x_2)-f_2(y,1-y)}{(y-x_1)^2}\right\} \\
&+\ \sum_{i=2}^{n-2}\int_0^{x_i+x_{i+1}} dy\frac{(x_i+y)(x_i+2x_{i+1}-y)}{4(x_i-y)^2\sqrt{x_ix_{i+1}y(x_i+x_{i+1}-y)}}\{f_n(x_1,\ldots,x_n) \\
&\quad -f_n(x_1,\ldots,x_{i-1},y,x_i+x_{i+1}-y,\ldots,x_n)\} + \frac{\pi}{4\sqrt{x_ix_{i+1}}}f_n \qquad (25) \\
&+\ \int_0^{x_{n-1}+x_n} dy\frac{x_n+2x_{n-1}-y}{2(x_n-y)^2\sqrt{x_{n-1}(x_n+x_{n-1}-y)}}\{f_n(x_1,\ldots,x_n)
\end{aligned}
$$

$$
-f_n(x_1,\ldots,x_n+x_{n-1}-y,y)\} + \frac{1}{x_n}\left(\sqrt{1+\frac{x_n}{x_{n-1}}}-1\right) f_n
$$

$$
\begin{aligned}
+ \quad & \int_0^{x_1+x_2} dy \frac{x_1+2x_2-y}{2(x_1-y)^2\sqrt{x_2(x_1+x_2-y)}}\{f_n(x_1,\ldots,x_n) \\
& -f_n(y,x_1+x_2-y,\ldots,x_n)\} + \frac{1}{x_1}\left(\sqrt{1+\frac{x_1}{x_2}}-1\right) f_n \ .
\end{aligned}
$$

The longitudinally induced g self-energy for fermions and gluons has a linear divergent piece which has been cancelled against the poles in y in each of the above integrals (curly brackets). These represent the x^+-instantaneous Coulomb exchange of A_+ quanta between partons, which diverges for zero exchanged momentum. The self-energy of the gluon also contains a logarithmic divergence which may be absorbed by the bare mass m_0 to leave renormalised mass m_B. The way we have written the two-body Coulomb kernels above means that certain finite parts are left over after cancellation of divergences and renormalisation of masses, i.e. we identify the following static interactions between two partons depending upon the statistics and momentum fraction of each

$$
\mathrm{quark}(x_1), \mathrm{gluon}(x_2) \qquad \frac{1}{x_1}\left(\sqrt{1+\frac{x_1}{x_2}}-1\right) \tag{26}
$$

$$
\mathrm{gluon}(x_i), \mathrm{gluon}(x_{i+1}) \qquad \frac{\pi}{4\sqrt{x_i x_{i+1}}} \tag{27}
$$

$$
\mathrm{quark, anti-quark} \qquad 0 \ . \tag{28}
$$

Since these are interactions between partons with contracted colour indices, it is useful to interpret them as groundstate energy of the flux line between partons. The Coulomb integrals describe the longitudinal excitations of the flux line in each case. For example, with no gluons ($n=2$) one recovers t'Hooft's model [1], the zero groundstate energy (28) resulting in a massless meson when $m \to 0$. When $m_B \to m \to 0$, a good approximation to the groundstate for each n is given by the phase space wavefunction ψ_n, defined by the ansatz $f_{n'} = \mathrm{const.}\delta_{nn'}$. The integrals shown and in fact all other longitudinal processes neglected in (25) vanish for this ansatz. For $n-2>0$ gluons one finds that M^2 is diagonal in the sub-basis of ψ_n's

$$
M^2|\psi_n> = g^2 N \left(\frac{((n-2)^2-1)\pi}{4} + \frac{4(n-1)\log 2}{\pi}\right)|\psi_n> , \tag{29}
$$

where the first term comes from gluon-gluon (27) and the second term from quark-gluon (26) flux energy. For small n we found that this formula typically differs from a numerical solution by about 10%. Each ψ_n is 2^n-fold degenerate in helicities and forms a groundstate on top of which there exists a discrete trajectory of Coulomb excitations just as for $n=2$ [1]. For high enough M however, this picture appears to break down since the processes which mix sectors of different n,

neglected in (25), become important [3]. But we emphasize that the non-zero longitudinal flux-line energies attached to gluons (27)(26) will tend to suppress their pair production, by whatever process, even when they are massless.

3.2 Transverse.

To a first approximation, the effect of s and t terms is to lift some of the helicity degeneracy. However they also mix sectors of different n and lead to new divergences. The t self-energy simply renormalises m_0. To illustrate the cancellation by the s self-energy we give below the integral equations for the $h = 0$ sector restricting Fock space to at most one gluon, the non-zero coefficients being f_{+-}, f_{-+}, f_{+-+}, and f_{-+-} in this case.

$$\begin{aligned} M^2 f_{+-}(x_1, x_2) &= m^2 \left(\frac{1}{x_1} + \frac{1}{x_2}\right) f_{+-}(x_1, x_2) \\ &+ \frac{s^2 g^2 N_c}{\pi} \int_0^\infty \frac{dy}{y} \left\{ \frac{1}{(x_1 - y)} + \frac{1}{(y + x_2)} \right\} f_{+-}(x_1, x_2) \\ &- msg\sqrt{\frac{N_c}{\pi}} \int_0^{x_1} \frac{dy}{\sqrt{y}} \left(\frac{1}{x_1 - y} - \frac{1}{x_1}\right) f_{-+-}(x_1 - y, y, x_2) \\ &- msg\sqrt{\frac{N_c}{\pi}} \int_0^{x_2} \frac{dy}{\sqrt{y}} \left(\frac{1}{x_2 - y} - \frac{1}{x_2}\right) f_{+-+}(x_1, y, x_2 - y) + \ldots \ , \end{aligned} \tag{30}$$

$$\begin{aligned} M^2 f_{-+-}(x_1, x_2, x_3) &= m^2 \left(\frac{1}{x_1} + \frac{1}{x_3}\right) f_{-+-}(x_1, x_2, x_3) \\ &- msg\sqrt{\frac{N_c}{\pi}} \frac{1}{\sqrt{x_2}} \left(\frac{1}{x_1} - \frac{1}{x_2 + x_1}\right) f_{+-}(x_1 + x_2, x_3) \\ &+ msg\sqrt{\frac{N_c}{\pi}} \frac{1}{\sqrt{x_2}} \left(\frac{1}{x_3} - \frac{1}{x_2 + x_3}\right) f_{-+}(x_1, x_2 + x_3) \\ &+ \frac{s^2 g^2 N_c}{\pi} \int_0^{x_2 + x_3} dy \frac{f_{-+-}(x_1, y, x_3 + x_2 - y)}{(x_2 + x_3)\sqrt{x_2 y}} \\ &+ \frac{s^2 g^2 N_c}{\pi} \int_0^{x_1 + x_2} dy \frac{f_{-+-}(x_1 + x_2 - y, y, x_3)}{(x_2 + x_1)\sqrt{x_2 y}} + \ldots \end{aligned} \tag{31}$$

and the same equations with $\{+ \leftrightarrow -, s \leftrightarrow -s\}$. Ellipses indicate purely longitudinal processes, dealt with in the previous subsection. The self-energy terms (curly brackets) together with the following two integrals in (30) appear divergent (they are). To see that the divergences cancel, rewrite the f_{-+-} integral singular at anti-quark momentum $x_1 - y = 0$ as

$$- msg\sqrt{\frac{N_c}{\pi}} \quad \left\{ \int_0^{x_1} \frac{dy}{\sqrt{x_1}(x_1 - y)} \left[f_{-+-}(x_1 - y, y, x_2) - f_{-+-}(0, x_1, x_2) \right] \right.$$

$$-\int_0^{x_1} \frac{dy}{x_1(\sqrt{x_1}+\sqrt{y})} f_{-+-}(x_1-y, y, x_2) + \frac{f_{-+-}(0,x_1,x_2)}{\sqrt{x_1}} \int_0^{x_1} \frac{dy}{x_1-y} \Bigg\} \quad (32)$$

to isolate a logarithmic divergence in the last integral of (32). From the behaviour of (31) as $x_1 \to 0$ one deduces the relation

$$f_{-+-}(0, x_2, x_3) = \frac{sg}{m}\sqrt{\frac{N_c}{\pi}} \frac{f_{+-}(x_2, x_3)}{\sqrt{x_2}}, \qquad (33)$$

to cancel the $\frac{1}{x_1}$ singularity (the longitudinal processes don't contribute at this order), and therefore the divergent term in (32) is

$$-\frac{s^2 g^2 N_c}{\pi} \int_0^{x_1} \frac{dy}{x_1(x_1-y)} \, f_{+-}(x_1, x_2), \qquad (34)$$

which is readily seen to cancel the anti-quark s self-energy divergence. The same argument goes through for the quark and other helicity. Such cancelations may be understood also through 1-loop light-cone perturbation theory [6]. In fact the full integral equations imply the general endpoint conditions for quarks:

$$f_{\alpha\beta\ldots\gamma\mp\pm}(x_1, \ldots, x_{n-2}, x_{n-1}, 0) = \pm\frac{sg}{m}\sqrt{\frac{N_c}{x_{n-1}\pi}} f_{\alpha\beta\ldots\gamma\mp}(x_1, \ldots, x_{n-2}, x_{n-1}) \qquad (35)$$

for all helicities, with a similar relation for anti-quarks. These may be used to show in exactly the same way the cancelation of divergent integrals due to zero momentum (anti)quarks for every equation i.e. involving an arbitrary number of gluons, since only the gluon neighbouring the quark in the large N_c colour contraction is involved.[3] Since no other divergent integrals arise, the boundstate equations are each finite. We offer numerical evidence for this finiteness in the next section. There is a possibility of divergence due to there being an infinite number of equations however. In fact this is to be expected for sufficiently large coupling constants from the the well-known divergence of large-N_c planar perturbation theory, which was studied in the light-front Hamiltonian formalism in refs.[2, 3].

4 Numerical Solutions.

The full boundstate equations may be solved numerically by discretizing the momentum fractions as $x = m/K$ for integers $\{m, K\}$ with $0 < m < K$, then extrapolating to $K = \infty$ [5]. Here we will exhibit results for one particular set of parameters: $m_B = 0$, $m^2 = 0.8$, $s = 0.5$, $t = 0.15$, $g^2 N_c = \pi$; the investigation of the full parameter space is left for future work. A tractable problem is only

[3] Restricting to no more than $n-2$ gluons, the s self-energies are omitted for Fock states with $n-2$ gluons.

obtained by further truncating the Fock space by hand. We therefore extrapolated to $K=\infty$ the discretised problem with at most two transverse gluons, and checked for finite K that the error in restricting the number of gluons was not large for the set of couplings used. This restriction is the simplest such that all terms in P^- (11) are used. Since the problem at finite K is equivalent to finite matrix diagonalisation of the M^2 operator [5], we employed a Lanczos algorithm whose implementation has been described elsewhere [2]. This algorithm is an iterative scheme which requires an input wavefunction close to an eigenfunction Ψ if convergence is to be good. Since the algorithm also preserves the symmetries of the theory, a sensible input will consist of a couple of valence quarks $\bar{q}q$ which transform in a definite way under symmetries of the discretised theory.

At finite K the symmetry C induced from charge conjugation

$$C: \; a_{\pm ij} \rightarrow -a_{\pm ji} \; , \; b_{\pm i} \leftrightarrow d_{\pm i} \tag{36}$$

remains exact, although parity $x^1 \rightarrow -x^1$ does not. In view of the difficulties in measuring true parity, at this stage we content ourselves with a classification according to the exact finite-K symmetry of the momentum fraction in the $\bar{q}q$ sector of Fock space

$$P_1 \; : \; x \leftrightarrow 1-x \; . \tag{37}$$

In the continuum theory this P_1-parity is equivalent to true parity in the limit of on-shell partons [9]. We take quarks and anti-quarks to have opposite intrinsic P_1 by analogy with true parity. We consider here the following valence combinations, classified by $|h|^{P_1 C}$:

$$0^{-+} \; : \quad d^\dagger_{+i}(x) b^\dagger_{-i}(1-x) - d^\dagger_{-i}(x) b^\dagger_{+i}(1-x)|0> \tag{38}$$

$$0^{--} \; : \quad d^\dagger_{+i}(x) b^\dagger_{-i}(1-x) + d^\dagger_{-i}(x) b^\dagger_{+i}(1-x)|0> \tag{39}$$

$$1^{--} \; : \quad d^\dagger_{\pm i}(x) b^\dagger_{\pm i}(1-x)|0> \; . \tag{40}$$

These couple strongly to the groundstate in each symmetry sector, and are evidently analogues of the J_z components of the π (or η') and ρ mesons, if we identify $h \equiv J_z$. Hybrids appear as excited $(*)$ states of them.

From the eigenfunctions Ψ one may compute unpolarized and polarized structure functions

$$Q(x) =< b^\dagger_{\alpha i} b_{\alpha i}(x) + d^\dagger_{\alpha i} d_{\alpha i}(x) > , \quad \Delta Q(x) =< \tfrac{1}{2} \textstyle\sum_\alpha \mathrm{sgn}(\alpha)[b^\dagger_{\alpha i} b_{\alpha i}(x) + d^\dagger_{\alpha i} d_{\alpha i}(x)] > \tag{41}$$

$$G(x) =< a^\dagger_{\alpha ij} a_{\alpha ij}(x) > , \qquad \Delta G(x) =< \textstyle\sum_\alpha \mathrm{sgn}\,(\alpha) a^\dagger_{\alpha ij} a_{\alpha ij}(x) > \tag{42}$$

which trivially satisfy momentum and helicity sum rules

$$\int_0^1 dx \; x[Q(x)+G(x)] \; = \; 1 \tag{43}$$

$$\int_0^1 dx \; [\Delta Q(x) + \Delta G(x)] \; = \; h \; . \tag{44}$$

It is also useful to define the integrated quantities $n_g = \int G(x)$, $\Delta n_q = \int \Delta Q(x)$, and the polarization asymmetries $A_g(x) = \Delta G/G$, $A_q(x) = \Delta Q/Q$. Numerical results for the chosen parameters are displayed in Table 1 and Figures 1 and 2. There is clear convergence of M^2, with 0^{-+} lightest (a robust feature). The lightest 'hybrid' occurs in the 1^{--} sector, suggesting that the hybrid ρ would be lightest. The Q and G of 0^{--} and 1^{--} turn out to be very similar to those of 0^{-+} plotted. G is small relative to Q due to the rather small s employed, neccesary to be sure of convergence. Note that even though the quarks are massive, Q does not vanish at $x \to 0$, as a result of eq.(33). Both the momentum and helicity of the 1^{--} are due to gluons at the level of about 10%. The helicity asymmetries (fig. 2) also support this helicity-momentum correlation, showing complete helicity alignment of the parton as $x \to 1$, while totally disordered polarization at small x.

5 Conclusions.

The 1+1-dimensional reduced model of light-front large-N_c QCD forms an interesting extension of 't Hooft's original model for two-dimensional mesons to include gluon degrees of freedom and helicity. We have performed light-front quantization of the normal modes and showed how divergences in the infinite set of coupled boundstate equations are absorbed by self-energies. A preliminary analysis of the meson boundstates was given and it is evidently desirable to make this more comprehensive, varying the parameters and in particular checking the true parity of states, which requires a more efficient computer/code. The way is now open to investigate non-perturbatively the behaviour of scattering amplitudes involving the mesons and glueballs constructed here and in ref.[2] within the $1/N_c$ expansion. It will be interesting to compare them with the hight-enrgy scattering processes of the four-dimensional world.

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FIGURE AND TABLE CAPTIONS

Figure 1. Quark and gluon momentum fraction distributions for 0^{-+} and 1_*^{--}; Solid lines Q; Broken lines G.

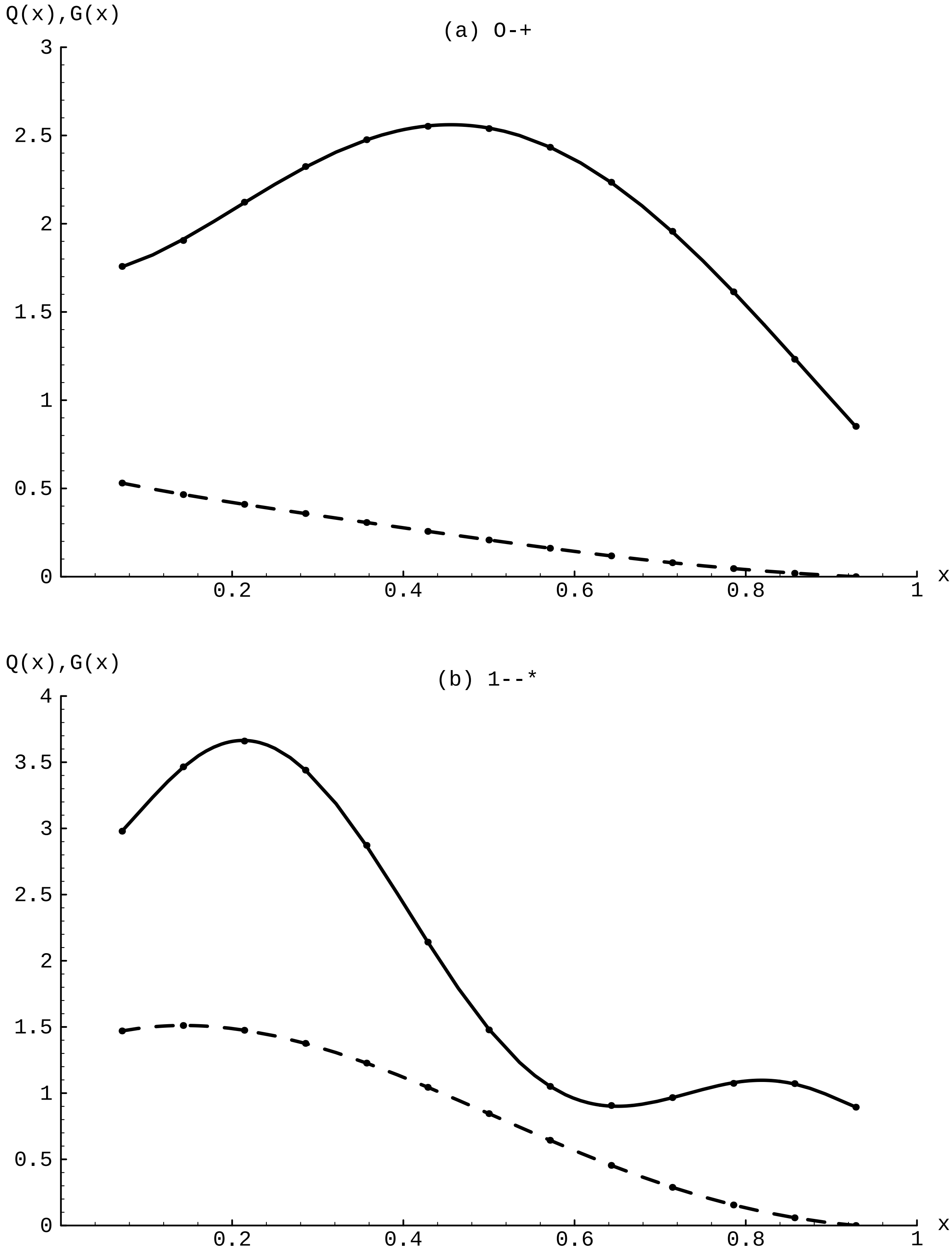
Figure 2. Quark and gluon helicity asymmetries for 1^{--}; solid line A_q; broken line A_g.

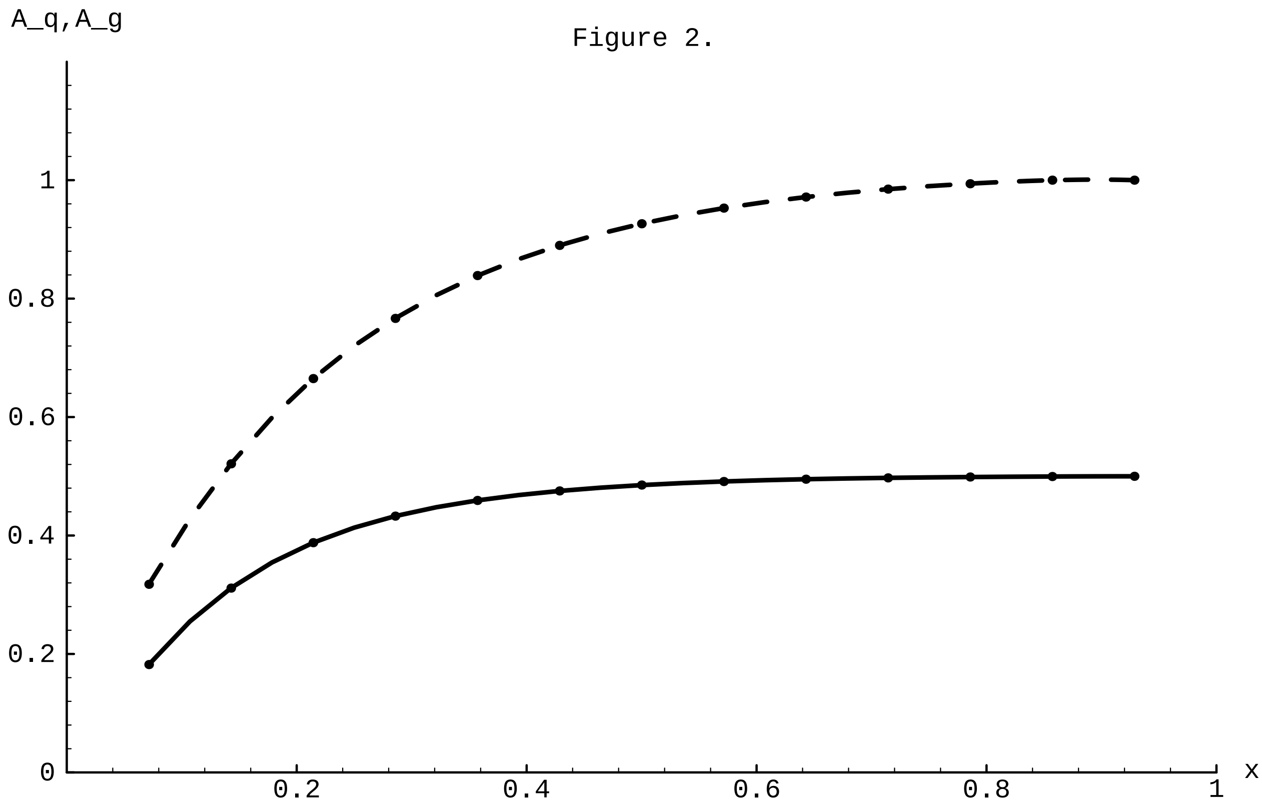
Table 1. Extrapolation of M^2 in cutoff K. n_g and Δn_q are quoted at $K = 14$.

Table.1

$M^2(K)$	0^{-+}	1^{--}	0^{--}	1_*^{--}
$M^2(12)$	5.3151	5.8393	6.2652	12.2958
$M^2(13)$	5.3812	5.9243	6.3599	12.5327
$M^2(14)$	5.4395	5.9992	6.4428	12.7455
$M^2(15)$	5.4913	6.0655	6.5158	12.9412
$M^2(16)$	5.5374	6.1247	6.5805	13.1229
$M^2(17)$	5.5789	–	–	–
$M^2(\infty)$	6.3	7.1	7.6	16.9
n_g	0.23	0.17	0.08	0.81
Δn_q	0	0.89	0	0.42

Figure 1.


Q(x),G(x)
(a) 0-+
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
x
Q(x),G(x)
(b) 1--*
4
3.5
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
x


A_q,A_g
Figure 2.
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x